The Complexity of Star Colouring and its Relatives

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Abstract

We study the complexity of three graph colouring variants–star colouring, restricted star colouring (abbr. rs colouring), and acyclic colouring–in relation to the maximum degree of the graph. For $k \in \mathbb{N}$, the problem k-Star Colourability takes a graph G as input, and asks whether G admits a star colouring with at most k colours. The problems k-RS Colourability and k-Acyclic Colourability are defined likewise. Consider a fixed integer $k \geq 3$, and the following questions.

1. What is the least integer d such that k-Star Colourability in graphs of maximum degree d is NP-complete?
2. What is the least integer d such that k-Star Colourability in d-regular graphs is NP-complete?
3. What is the highest integer d such that k-Star Colourability in d-regular graphs is NP-complete?

Let $L_s^{(k)}$, $\widetilde{L}_s^{(k)}$ and $H_s^{(k)}$ denote the answers to Questions 1, 2 and 3, respectively. For similar questions on rs colouring (resp. acyclic colouring), let $L_{rs}^{(k)}$, $\widetilde{L}_{rs}^{(k)}$ and $H_{rs}^{(k)}$ (resp. $L_a^{(k)}$, $\widetilde{L}_a^{(k)}$ and $H_a^{(k)}$) denote the answers. From reductions in the literature, it follows that $L_s^{(k)} \leq k(k-1+\lceil\sqrt{k}\,\rceil)$ and $L_a^{(k)} \leq k(k-1+\lceil\sqrt{k}\,\rceil)$. We prove linear upper bounds on $L_s^{(k)}$, $L_{rs}^{(k)}$ and $L_a^{(k)}$: (i) $L_s^{(3)} = 3$ and $L_s^{(k)} \leq k$ for $k \geq 4$, (ii) $L_{rs}^{(3)} = 3$ and $L_{rs}^{(k)} \leq k-1$ for $k \geq 4$, and (iii) $L_a^{(k)} \leq k+1$. We also show that (i) for $k = 5$ and $k \geq 7$, $\widetilde{L}_s^{(k)} = L_s^{(k)}$ and $H_s^{(k)} \leq 2k-4$, (ii) for $k \geq 4$, $\widetilde{L}_{rs}^{(k)} = L_{rs}^{(k)}$ and $H_{rs}^{(k)} = k-1$; and (iii) for $k \geq 4$, $\widetilde{L}_a^{(k)} = L_a^{(k)}$ and $H_a^{(k)} = 2k-3$. We conjecture that $H_s^{(k)} = 2k-4$ for $k \geq 4$, and prove this conjecture for $k = 4$. In addition, we prove NP-completeness results on star colouring, rs colouring and acyclic colouring in well-known graph classes such as planar graphs and bipartite graphs; e.g.: 3-Star Colourability is known to be NP-complete in (i) planar bipartite graphs, (ii) graphs of maximum degree 4, and (iii) graphs of arbirtarily large girth; we show that it is NP-complete in the intersection of the three classes.