Synopsis report of

ELIMINATION OF OVERFLOW OSCILLATIONS IN NONLINEAR DISCRETE-TIME STATE-DELAYED SYSTEMS IN THE PRESENCE OF EXTERNAL DISTURBANCE

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1 Introduction

Digital signal processing in most of the applications are intended to be linear operations. However, the processing in discrete-time systems with finite wordlength implies that the linearity can be reached to a certain level and the deviation from the linear operation can be minimized by selecting longer wordlength. Yet there may be finite wordlength effects due to which nonlinearities may occur that drive the linear discrete-time systems into nonlinear discrete-time systems. An essential issue in the field of nonlinear systems is the qualitative analysis, especially the stability analysis. In this research work, a particular class of nonlinear systems, digital filters with overflow nonlinearities are considered to investigate stability properties. Digital filter is a discrete-time system that works according to a designed algorithm to modify certain parameters of discrete-time signals. Digital filters have become increasingly popular in the last few decades due to rapid drop in cost. As an important component of modern electronics, digital filters' widespread applications include mobile phones, automobiles, biomedical, robotics and many others. Before the digital filters, analog filters were the only option. However, reliability, repeatability, flexibility and low cost of the digital systems have made their analog counter parts less common.

1.1 Digital Filters with Finite Wordlength Nonlinearities, External Disturbance and State-delay

In the implementation of digital filters, only finite amount of precision can be allocated to each number, and so arithmetic computations must be modified after a finite number of digits. In every digital filter, arithmetic operations such as additions and multiplications take place which in most cases increase wordlength required for the operation. As a result, a wordlength reduction strategy is required to avoid the signals from accumulating an ever-increasing wordlength [1–3]. This wordlength reduction can be done by using quantization and overflow correction. These operations may produce nonlinearities in the systems. Overflow methods modify most significant and least significant bits whereas quantization techniques influence only least significant bits of a fixed-point number. So, overflow nonlinearities cause severe distortion as compared to quantization nonlinearities. The type of nonlinearities in the digital filters depends on the kind of arithmetic operation used to eliminate overflow occurred from computation. The commonly used overflow methods are saturation, zeroing, two's complement and triangular. As it is evident from literature that the saturation overflow arithmetic produces the smallest deviation from the linear operation [2], stability analysis of digital filters with saturation arithmetic has been considered as an important research topic [1, 4–11].

The presence of time-delays in the dynamical systems is another source for instability [12, 13]. These kinds of delays generally appear during the modeling of physical systems due to transport lags, measurement lags and signal transmission with finite speeds . A few research works have been carried out to investigate stability of state-delayed digital filters with finite wordlength nonlinearities [9, 11, 14].

On another research front, it has been recognized that systems performance may be degraded in the presence of external disturbance [15, 16]. The disturbance may be encountered in the higher-order digital filters when implemented using the lower-order digital filters to prevent finite wordlength effects [17]. The existence of external interference in the higher-order digital filters may cause the implemented filter to work inappropriately [15, 16, 18, 19].

2 Motivation and Objectives

It is observed from Section 1 that digital filters implemented on finite wordlength machines may become nonlinear digital filters due to presence quantization and overflow nonlinearities. Many stability techniques have been proposed to ensure stability of the digital filters with finite wordlength nonlinearities and external disturbance [8, 10, 18]. The approaches like H_∞ performance index, input-output-state-stability and input-state-stability have been used to test robust stability of the digital filters with overflow nonlinearities and external disturbance. It is known that the delay in the systems may lead to instability [12, 13]. However, most of the works in literature have not considered the effects of delay in the interfered digital filters employing overflow arithmetic. Thus, conditions to examine stability of the state-delayed digital filters with saturation nonlinearities and external disturbance are not available in the literature. This is the first motivation of this thesis.

Presently, researchers are investigating stability of digital filters to establish conditions which ensure non-existence of limit-cycle oscillations. It is well understood that a digital filter is free from limit-cycles when its null solution is asymptotically stable [2, 3]. Thus, most of the works in the literature have been presented to confirm the stability of digital filters with finite wordlength nonlinearities. However, it is observed that stability conditions in the existing methods may be conservative and thus, there is some scope to develop less restrictive criteria which may improve stability region of the digital filters. This is the second motivation of this thesis.

Motivated by these information, in this thesis, a few approaches are proposed to develop stability conditions for the digital filters with overflow nonlinearities.

3 Contribution of the Thesis

The following are the significant contributions of this thesis:

- A method to examine stability and H_∞ performance of the state-delayed interfered digital filters employing saturation overflow arithmetic is developed using Lyapunov energy function and sector-based characterization. The established conditions guarantee exponential stability of the digital filters without external disturbance and suppress effects of external disturbance using H_∞ performance norm.

- An improved approach is developed to investigate stability of the discrete-time state-delayed systems with saturation nonlinearities and external interference. With the help of better characterization of saturation arithmetic, a sufficient condition is formulated for the stability of interfered digital filters with state-delay and saturation nonlinearities. It is demonstrated that the established stability criterion is less strict than the existing stability conditions.

- By making use of the system transformation and an appropriate Lyapunov functional, stability study is conducted for fixed-point digital filters with time-varying delay, saturation nonlinearities and external interference. This approach, first time, reports a criterion to ensure H_∞ performance of the digital filters with disturbance, saturation nonlinearities and time-varying delay. Further, a stability condition is developed for the interference free systems with saturation arithmetic and null state-delay. The presented condition is demonstrated to be less conservative than the existing results.

- The problem of delay-dependent stability investigation of digital filters with disturbance and saturation overflow nonlinearities is extended for generalized overflow nonlinearities. In this work, two new lemmas related to characterization of overflow nonlinearities are developed and a Lyapunov functional with more delay information is constructed. Based on the established lemmas and the Lyapunov functional, a stability criterion is proposed for the digital filters with time-varying delay and overflow nonlinearities. The presented condition is shown to be more relaxed and computationally less demanding than recent existing criteria. Further, effect of external disturbance in the digital filters with time-varying delay and generalized overflow nonlinearities is studied with the help of H_∞ performance.

Notations: The symbols used in this work are standard. The notation $\boldsymbol{R}^{n\times n}$ denotes $n \times n$ real matrices. Positive (negative) definite symmetric matrix is represented by $\boldsymbol{G} > \mathbf{0}$ ($\boldsymbol{G} < \mathbf{0}$). The unity matrix is designated as $\boldsymbol{I}$ and null matrix or vector described as $\mathbf{0}$. The symbol '*' represents symmetric entries in a symmetric matrix. The symbols $\lambda_{min}(\boldsymbol{G})$ and $\lambda_{max}(\boldsymbol{G})$ indicate minimum and maximum eigenvalues of the matrix $\boldsymbol{G}$, respectively.

4 Description of the Proposed Work

The stability of digital filters with overflow nonlinearities, external disturbance, constant/time-varying state-delay is investigated in this thesis. The proposed techniques utilize system information in a greater detail, employs better characterization of overflow nonlinearities and construct the Lyapunov functional with better system information.

4.1 Realization of Overflow Oscillations Free Interfered Nonlinear Discrete-time Systems with State-delay and Saturation Nonlinearities

In this work, H_∞ performance and exponential stability of state-delayed interfered digital filters with saturation nonlinearities are addressed.

4.1.1 System Description

The discrete-time system under consideration is given as:

$$\boldsymbol{x}(k+1) = \boldsymbol{f}\big(\boldsymbol{y}(k)\big) + \boldsymbol{w}(k), \tag{1a}$$

$$\boldsymbol{y}(k) = \boldsymbol{A}\boldsymbol{x}(k) + \sum_{i=1}^{m} \boldsymbol{A}_{d_i}\boldsymbol{x}(k-d_i), \tag{1b}$$

$$\boldsymbol{x}(k) = \boldsymbol{\phi}(k), \quad k \in [-d, 0], \tag{1c}$$

$$d = max\{d_1, d_2, ..., d_m\}, \tag{1d}$$

where $\boldsymbol{x}(k) \in \boldsymbol{R}^n$ denotes the state vector, $\boldsymbol{w}(k) \in \boldsymbol{R}^n$ describes the external disturbance vector, $\boldsymbol{y}(k) \in \boldsymbol{R}^n$ signifies the output vector and initial conditions are denoted as $\boldsymbol{\phi}(k) \in \boldsymbol{R}^n$. The matrices $\boldsymbol{A}_{d_i}(i = 1, 2, ..., m) \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$ are system matrices, $d_i > 0$ $(i = 1, 2, ..., m)$ is a positive scalar and m represents number of delayed states. The function $\boldsymbol{f}(.)$ representing saturation nonlinearity characteristics is defined as

$$f_i\big(y_i(k)\big) = \begin{cases} 1, & y_i(k) > 1 \\ y_i(k), & -1 \le y_i(k) \le 1, \quad i = 1, 2, ..., k. \\ -1, & y_i(k) < -1 \end{cases} \tag{2}$$

For the digital filter under consideration with the H_∞ performance $\gamma > 0$, the aim of this work is twofold which is as follows. Firstly, to determine a new exponential stability condition for the system (1)-(2) subject to

$$\sum_{k=0}^{\infty} \boldsymbol{x}^T(k)\boldsymbol{S}\boldsymbol{x}(k) < \gamma^2 \sum_{k=0}^{\infty} \boldsymbol{w}^T(k)\boldsymbol{w}(k) \tag{3}$$

with zero initial conditions for all $\boldsymbol{w}(k) \neq \boldsymbol{0}$, where the matrix $\boldsymbol{S}$ is a positive definite symmetric matrix. Secondly, to formulate a criterion for the system (1)-(2) with guaranteed exponential stability when external vanishes. To characterize saturation arithmetic, the following lemma is presented.

Lemma 1 *Let $\boldsymbol{D} = [d_{st}] \in \boldsymbol{R}^{n\times n}$ be a diagonal matrix with positive diagonal elements, then the following holds.*

$$\boldsymbol{f}^T\big(\boldsymbol{y}(k)\big)\boldsymbol{D}\big[\boldsymbol{y}(k) - \boldsymbol{f}\big(\boldsymbol{y}(k)\big)\big] + \big[\boldsymbol{y}(k) - \boldsymbol{f}\big(\boldsymbol{y}(k)\big)\big]^T\boldsymbol{D}\boldsymbol{f}\big(\boldsymbol{y}(k)\big) \ge 0. \tag{4}$$

A linear matrix inequality (LMI) criterion to guarantee exponential stability and H_∞ performance of the system (1)-(2) is presented in the next subsection.

4.1.2 Main Results

A stability condition is presented via the following theorem.

Theorem 1 *For given interference attenuation level $\gamma > 0$, the digital filter (1)-(2) is exponentially stable if there exist positive definite symmetric matrices $\boldsymbol{P} > \boldsymbol{0}$, $\boldsymbol{S} > \boldsymbol{0}$, $\boldsymbol{Q}_i > \boldsymbol{0}$ $(i = 1, 2, ..., m)$, a positive diagonal matrix $\boldsymbol{D}$ and a scalar $\delta > 0$ such that the following LMI holds true*

$$\begin{bmatrix} \delta\boldsymbol{A}^T\boldsymbol{A} + \sum_{i=1}^{m}\boldsymbol{Q}_i - \boldsymbol{P} + \boldsymbol{S} & \delta\boldsymbol{A}^T\boldsymbol{A}_{d_1} & \dots & \delta\boldsymbol{A}^T\boldsymbol{A}_{d_m} & \boldsymbol{A}^T\boldsymbol{D} & \boldsymbol{0} \\ * & \delta\boldsymbol{A}_{d_1}^T\boldsymbol{A}_{d_1} - \boldsymbol{Q}_1 & \dots & \delta\boldsymbol{A}_{d_1}^T\boldsymbol{A}_{d_m} & \boldsymbol{A}_{d_1}^T\boldsymbol{D} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \dots & \delta\boldsymbol{A}_{d_m}^T\boldsymbol{A}_{d_m} - \boldsymbol{Q}_m & \boldsymbol{A}_{d_m}^T\boldsymbol{D} & \boldsymbol{0} \\ * & * & \dots & * & \boldsymbol{P} - \delta\boldsymbol{I} - 2\boldsymbol{D} & \boldsymbol{P} \\ * & * & \dots & * & * & \boldsymbol{P} - \gamma^2\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}. \tag{5}$$

Remark 1 *Theorem 1 addresses stability of the externally disturbed digital filters subject to saturation nonlinearities and state-delay in order to provide a suitable condition that assures exponential stability with H_∞ performance γ.*

It is interesting to note that the influence of saturation nonlinearities on the stability of digital filters is accessed via (4). The criterion presented via Theorem 1 may have less degree of freedom because of structure of the matrix $\boldsymbol{D}$. This may lead to conservative results. In the next theorem, we present an improved stability condition for the externally interfered state-delayed digital filters with saturation nonlinearities by relaxing constraints in the matrix $\boldsymbol{D}$.

Theorem 2 *If there exist positive definite symmetric matrices $\boldsymbol{P} > \boldsymbol{0}$, $\boldsymbol{S} > \boldsymbol{0}$, $\boldsymbol{Q}_i > \boldsymbol{0}$ $(i = 1, 2, ..., m)$, a row diagonally dominant matrix $\boldsymbol{C}$ with positive diagonal elements such that*

$$\begin{bmatrix} \delta\boldsymbol{A}^T\boldsymbol{A} + \sum_{i=1}^{m}\boldsymbol{Q}_i - \boldsymbol{P} + \boldsymbol{S} & \delta\boldsymbol{A}^T\boldsymbol{A}_{d_1} & \dots & \delta\boldsymbol{A}^T\boldsymbol{A}_{d_m} & \boldsymbol{A}^T\boldsymbol{C} & \boldsymbol{0} \\ * & \delta\boldsymbol{A}_{d_1}^T\boldsymbol{A}_{d_1} - \boldsymbol{Q}_1 & \dots & \delta\boldsymbol{A}_{d_1}^T\boldsymbol{A}_{d_m} & \boldsymbol{A}_{d_1}^T\boldsymbol{C} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \dots & \delta\boldsymbol{A}_{d_m}^T\boldsymbol{A}_{d_m} - \boldsymbol{Q}_m & \boldsymbol{A}_{d_m}^T\boldsymbol{C} & \boldsymbol{0} \\ * & * & \dots & * & \boldsymbol{P} - \delta\boldsymbol{I} - \boldsymbol{C} - \boldsymbol{C}^T & \boldsymbol{P} \\ * & * & \dots & * & * & \boldsymbol{P} - \gamma^2\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}, \tag{6}$$

then the system (1)-(2) is exponentially stable with H_∞ performance γ.

Remark 2 *One may obtain Theorem 1 from Theorem 2 by replacing $\boldsymbol{C} = \boldsymbol{C}^T = \boldsymbol{D}$ in (6), where $\boldsymbol{C}$ is a diagonal dominant matrix and $\boldsymbol{D}$ is a diagonal matrix. As a result, Theorem 2 may be seen as an extended result of Theorem 1 for the interfered state-delayed digital filters that use saturation arithmetic.*

Remark 3 *Theorem 2 is obtained by characterizing (2) using $\big(\boldsymbol{f}^T\big(\boldsymbol{y}(k)\big)\boldsymbol{C}\big[\boldsymbol{y}(k) - \boldsymbol{f}\big(\boldsymbol{y}(k)\big)\big]\big)$. In comparison to Theorem 1, the non-diagonal entries of matrix $\boldsymbol{C}$ provide more degrees of freedom, allowing for a more stable region in the parameter space.*

The following numerical examples is considered to demonstrate superiority of Theorem 2 over Theorem 1. It is checked with the help of YALMIP parser [20] that criterion proposed via Theorem 1 fails to give a feasible solution for

$$\boldsymbol{A} = \begin{bmatrix} 1.2 & -1.5 \\ 0.1 & 0.5 \end{bmatrix}, \quad \boldsymbol{A}_{d_1} = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \quad \boldsymbol{w}(k) = \begin{bmatrix} cos(k) \\ sin(k) \end{bmatrix} \text{ and } \ m = 1. \tag{7}$$

However, it is found that Theorem 2 succeeds to provide a feasible solution. Therefore, Theorem 2 is less stringent stability condition than Theorem 1.

4.2 Stability Investigation of Externally Disturbed State-delayed Digital Filters Employing Saturation Arithmetic

In section 4.1, first time, stability criteria are reported to access influence of saturation overflow nonlinearities on the stability of state-delayed interfered digital filters via H_∞ approach. The presented results are sufficient criteria and hence, there is still plenty of opportunity to improve the current stability conditions to obtain better stability region.

4.2.1 Problem Formulation

In this work, H_∞ performance and stability of state-delayed digital filters with interference and saturation nonlinearities are examined. The aim of this approach is to develop a better stability criterion by improvising the characterization of saturation nonlinearities. The developed condition ensures limit-cycle free realization of state-delayed digital filters with disturbance and saturation nonlinearities, and satisfies the condition (3) with better attenuation level γ.

To characterize saturation arithmetic (2), the following lemma is given.

Lemma 2 *Let there exist scalars* $\tau_{pq} > 0, \ \ \omega_{pq} > 0 \ (p, q = 1, 2, ..., n \ \ (q \neq p)),$ *a matrix* $\boldsymbol{E} = [E_{pq}] \in \boldsymbol{R}^{n \times n}$ *such that*

$$E_{pp} = \sum_{p=1, q \neq p}^{n} (\tau_{pq} + \omega_{pq}), \qquad q = 1, 2, ..., n, \tag{8a}$$

$$E_{pq} = (\tau_{pq} - \omega_{pq}), \qquad p, q = 1, 2, ..., n \ \ (q \neq p), \tag{8b}$$

then

$$\left[\boldsymbol{y}^T(k)(\boldsymbol{E}^T + \boldsymbol{E})\boldsymbol{y}(k) - \boldsymbol{f}^T\big(\boldsymbol{y}(k)\big)\left(\boldsymbol{E}^T + \boldsymbol{E}\right)\boldsymbol{f}\big(\boldsymbol{y}(k)\big)\right] \geq 0, \tag{9}$$

where $\boldsymbol{y}(k)$ *is output of the system (1)-(2) and* $\boldsymbol{f}\big(\boldsymbol{y}(k)\big)$ *represents saturation operator.*

In the next subsection, an improved LMI based criterion for the system (1)-(2) is presented.

4.2.2 Main Results

A stability condition for the system (1)-(2) is proposed using Lyapunov function and Lemma 2 via Theorem 3. Further, Theorem 5 is used to offer a better stability criterion for the digital filters employing saturation arithmetic.

Theorem 3 *The system (1)-(2) is exponentially stable for given $\gamma > 0$, if there exist matrices $\boldsymbol{C} = [C_{st}] \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{E} = [E_{pq}] \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{Q}_i > \boldsymbol{0}$ $(i = 1, 2, ..., m)$, $\boldsymbol{S} > \boldsymbol{0}$ and $\boldsymbol{P} > \boldsymbol{0}$ such that*

$$\begin{bmatrix} \boldsymbol{\Theta}_{11} & \boldsymbol{A}^T(\boldsymbol{E}+\boldsymbol{E}^T)\boldsymbol{A}_{d_1} & \cdots & \boldsymbol{A}^T(\boldsymbol{E}+\boldsymbol{E}^T)\boldsymbol{A}_{d_m} & \boldsymbol{A}^T\boldsymbol{C} & \boldsymbol{0} \\ * & \boldsymbol{A}_{d_1}^T(\boldsymbol{E}+\boldsymbol{E}^T)\boldsymbol{A}_{d_1}-\boldsymbol{Q}_1 & \cdots & \boldsymbol{A}_{d_1}^T(\boldsymbol{E}+\boldsymbol{E}^T)\boldsymbol{A}_{d_m} & \boldsymbol{A}_{d_1}^T\boldsymbol{C} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & \boldsymbol{A}_{d_m}^T(\boldsymbol{E}+\boldsymbol{E}^T)\boldsymbol{A}_{d_m}-\boldsymbol{Q}_m & \boldsymbol{A}_{d_m}^T\boldsymbol{C} & \boldsymbol{0} \\ * & * & \cdots & * & \boldsymbol{P}-\boldsymbol{E}-\boldsymbol{E}^T-\boldsymbol{C}-\boldsymbol{C}^T & \boldsymbol{P} \\ * & * & \cdots & * & * & \boldsymbol{P}-\gamma^2\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}, \tag{10}$$

$$C_{ss} = \sum_{s=1,t\neq s}^{n} (\alpha_{st} + \beta_{st}), \quad t = 1, 2, ..., n, \tag{11a}$$

$$C_{st} = (\alpha_{st} - \beta_{st}), \quad \alpha_{st} > 0, \ \beta_{st} > 0, \quad s, t = 1, 2, ..., n \ (t \neq s), \tag{11b}$$

where

$$\boldsymbol{\Theta}_{11} = \boldsymbol{A}^T(\boldsymbol{E}+\boldsymbol{E}^T)\boldsymbol{A} + \sum_{i=1}^{m}\boldsymbol{Q}_i - \boldsymbol{P} + \boldsymbol{S}. \tag{12}$$

Remark 4 *It is worth to mention that Theorem 3 is identified as Theorem 2, which ensures stability of interfered digital filters with delay and saturation arithmetic in the situation $\boldsymbol{E} = \boldsymbol{E}^T = \frac{1}{2}\delta\boldsymbol{I}$, where δ is a positive scalar. Further, by fixing $\alpha_{st} = \beta_{st}$ and $\boldsymbol{E} = \boldsymbol{E}^T = \frac{1}{2}\delta\boldsymbol{I}$, Theorem 1 may be recovered from Theorem 3.*

For the system (1)-(2) with zero delay, the following result is established.

Theorem 4 *The system (1)-(2) with zero delay is exponentially stable for given $\gamma > 0$, if there exist matrices $\boldsymbol{C} = [C_{st}] \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{E} = [E_{pq}] \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{S} > \boldsymbol{0}$ and $\boldsymbol{P} > \boldsymbol{0}$ such that*

$$\begin{bmatrix} \boldsymbol{A}^T(\boldsymbol{E}+\boldsymbol{E}^T)\boldsymbol{A}-\boldsymbol{P}+\boldsymbol{S} & \boldsymbol{A}^T\boldsymbol{C} & \boldsymbol{0} \\ * & \boldsymbol{P}-\boldsymbol{E}-\boldsymbol{E}^T-\boldsymbol{C}-\boldsymbol{C}^T & \boldsymbol{P} \\ * & * & \boldsymbol{P}-\gamma^2\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}. \tag{13}$$

Remark 5 *Theorem 4 is the condition for overflow oscillation free realization of digital filters using saturation characteristics with disturbance.*

For the system (1)-(2) with null delay and zero disturbance, the following result is stated.

Theorem 5 *The delay-free system characterized by* (1)-(2) *with zero external disturbance is asymptotically stable, suppose there exist matrices $\boldsymbol{C} = [C_{st}] \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{E} = [E_{pq}] \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{P} > \boldsymbol{0}$ such that*

$$\begin{bmatrix} \boldsymbol{A}^T(\boldsymbol{E}+\boldsymbol{E}^T)\boldsymbol{A}-\boldsymbol{P} & \boldsymbol{A}^T\boldsymbol{C} \\ * & \boldsymbol{P}-\boldsymbol{E}-\boldsymbol{E}^T-\boldsymbol{C}-\boldsymbol{C}^T \end{bmatrix} < \boldsymbol{0}. \tag{14}$$

Remark 6 *For discrete-time systems employing saturation arithmetic, Theorem 5 provides an improved stability criterion. The criterion proposed in Theorem 5 is less stringent than the existing results [6, 7].*

It is worth to point out that Theorems 1 and 2 do not give any feasible solution for the system (1)-(2) with following values

$$\boldsymbol{A} = \begin{bmatrix} 1.1 & -0.5 \\ 0.5 & 0.01 \end{bmatrix}, \ \boldsymbol{A}_{d_1} = \begin{bmatrix} 0.1 & 0.001 \\ 0.001 & 0 \end{bmatrix}, \ \boldsymbol{w}(k) = \begin{bmatrix} cos(k) \\ sin(k) \end{bmatrix} \text{ and } \ m = 1. \tag{15}$$

However, it is found that stability condition given by Theorem 3 succeeds to provide a feasible solution. Thus, Theorem 3 is more relaxed than Theorems 1 and 2 reported for limit-cycle free interfered digital filters with saturation arithmetic and state-delay.

4.3 Delay-dependent Stability Criteria for Interfered Digital Filters with Time-varying Delay and Saturation Nonlinearities

The H_∞ performance and exponential stability of state-delayed interfered digital filters with saturation nonlinearities have been addressed in the previous works. But, Theorems 1, 2 and 3 are not applicable to test stability of the discrete-time systems with time-varying state-delay. This work develops a new criterion to ensure exponential stability of the digital filters with saturation nonlinearities, external disturbance and time-varying delay.

4.3.1 System Model and Problem Formulation

The digital filter with external disturbance, saturation nonlinearities and state-delay is given by

$$\boldsymbol{x}(k+1) = \boldsymbol{f}\big(\boldsymbol{y}(k)\big) + \boldsymbol{w}(k), \tag{16a}$$

$$\boldsymbol{y}(k) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{A}_d\boldsymbol{x}\big(k - h(k)\big), \tag{16b}$$

$$\boldsymbol{x}(k) = \boldsymbol{\phi}(k), \qquad k \in [-h_2, 0], \tag{16c}$$

where time-varying delay $h(k)$ satisfies $h_1 \leq h(k) \leq h_2$ with h_1 and h_2 are being lower and upper delay bounds of the system (16), respectively. The main goal of this work is to develop a criterion for the discrete-time system represented by (16) and (2) to ensure exponential stability when external disturbance disappears and guarantee H_∞ performance of the system in the presence of external disturbance.

4.3.2 Main Results

A stability criterion for the system characterized by (16) and (2) is presented via the following theorem.

Theorem 6 *For a given convergence rate $\alpha > 0$ and an attenuation level $\gamma > 0$, the system given by* (16) *and* (2) *satisfying $h_1 \leq h(k) \leq h_2$ is exponentially stable, if there exist $\boldsymbol{P} > \boldsymbol{0}$, $\boldsymbol{S} > \boldsymbol{0}$, $\boldsymbol{Q}_i > \boldsymbol{0}$ $(i = 1, 2, 3)$, $\boldsymbol{R}_i > \boldsymbol{0}$ $(i = 1, 2)$, scalars $\nu_{ij}^{(1)}$, $\varsigma_{ij}^{(1)}$, $\nu_{ij}^{(2)}$, $\varsigma_{ij}^{(2)}$ $(i, j = 1, 2, ..., n(i \neq j))$ and a matrix $\boldsymbol{L}$ such that*

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}_1 + \boldsymbol{\Pi}_2 + \boldsymbol{\Pi}_3 + \boldsymbol{\Pi}_4 + \boldsymbol{\Pi}_4^T + \boldsymbol{\Pi}_5 < \boldsymbol{0}, \tag{17}$$

$$\hat{\boldsymbol{\Pi}} = \begin{bmatrix} \boldsymbol{R}_2 & \boldsymbol{L} \\ \boldsymbol{L}^T & \boldsymbol{R}_2 \end{bmatrix} > \boldsymbol{0}, \tag{18}$$

where

$$\boldsymbol{\Pi}_1 = \boldsymbol{g}_1^T\boldsymbol{P}\boldsymbol{g}_1 + \boldsymbol{g}_1^T\boldsymbol{P}\boldsymbol{g}_6 + \boldsymbol{g}_6^T\boldsymbol{P}\boldsymbol{g}_1 + \boldsymbol{g}_6^T\boldsymbol{P}\boldsymbol{g}_6 - \gamma^2\boldsymbol{g}_6^T\boldsymbol{g}_6 + \boldsymbol{\Xi}_0^T(h_1^2\boldsymbol{R}_1 + h_{12}^2\boldsymbol{R}_2)\boldsymbol{\Xi}_0 - \boldsymbol{g}_2^T(\boldsymbol{P} - \boldsymbol{S})\boldsymbol{g}_2$$

$$\boldsymbol{\Pi}_2 = \boldsymbol{g}_2^T\left(\boldsymbol{Q}_1 + (h_{12}+1)\boldsymbol{Q}_3\right)\boldsymbol{g}_2 - \boldsymbol{g}_3^T\left(\boldsymbol{Q}_1 - \boldsymbol{Q}_2\right)\boldsymbol{g}_3 - \boldsymbol{g}_5^T\boldsymbol{Q}_2\boldsymbol{g}_5 - \boldsymbol{g}_4^T\boldsymbol{Q}_3\boldsymbol{g}_4$$

$$\boldsymbol{\Pi}_3 = \left(e^{\alpha}\boldsymbol{A}\boldsymbol{g}_2 + e^{\alpha[h(k)+1]}\boldsymbol{A}_d\boldsymbol{g}_4\right)^T\left(\boldsymbol{\Lambda}^{(1)^T} + \boldsymbol{\Lambda}^{(1)}\right)\left(e^{\alpha}\boldsymbol{A}\boldsymbol{g}_2 + e^{\alpha[h(k)+1]}\boldsymbol{A}_d\boldsymbol{g}_4\right) - \boldsymbol{g}_1^T\left(\boldsymbol{\Lambda}^{(1)^T} + \boldsymbol{\Lambda}^{(1)}\right)\boldsymbol{g}_1$$

$$\boldsymbol{\Pi}_4 = \boldsymbol{g}_1^T\boldsymbol{\Lambda}^{(2)^T}[e^{\alpha}\boldsymbol{A}\boldsymbol{g}_2 + e^{\alpha[h(\gamma)+1]}\boldsymbol{A}_d\boldsymbol{g}_4 - \boldsymbol{g}_1]$$

$$\begin{aligned}
&\mathbf{\Pi}_5 = -\mathbf{\Xi}_1^T \boldsymbol{R}_1 \mathbf{\Xi}_1 - \begin{bmatrix} \mathbf{\Xi}_2 \\ \mathbf{\Xi}_3 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{R}_2 & \boldsymbol{L} \\ \boldsymbol{L}^T & \boldsymbol{R}_2 \end{bmatrix} \begin{bmatrix} \mathbf{\Xi}_2 \\ \mathbf{\Xi}_3 \end{bmatrix} \\
&\mathbf{\Xi}_0 = \boldsymbol{g}_1 + \boldsymbol{g}_6 - \boldsymbol{g}_2, \quad \mathbf{\Xi}_i = \boldsymbol{g}_{i+1} - \boldsymbol{g}_{i+2} \ (i = 1, 2, 3) \\
&\boldsymbol{g}_i = [\mathbf{0}_{n\times(i-1)n} \quad \boldsymbol{I}_n \quad \mathbf{0}_{n\times(6-i)n}] \ (i = 1, 2, ..., 6) \\
&h_{12} = h_2 - h_1,
\end{aligned}$$

and matrix $\mathbf{\Lambda}^{(q)} = [\Lambda_{ij}^{(q)}] \in \boldsymbol{R}^{n\times n}$ $(q = 1, 2)$ *is structured as*

$$\Lambda_{ii}^{(q)} = \sum_{i=1, i\neq j}^{n} \nu_{ij}^{(q)} + \varsigma_{ij}^{(q)}, \qquad j = 1, 2, ..., n, \tag{20a}$$

$$\Lambda_{ij}^{(q)} = \nu_{ij}^{(q)} - \varsigma_{ij}^{(q)}, \qquad i, j = 1, 2, ..., n \ (i \neq j), \tag{20b}$$

$$\nu_{ij}^{(q)} > 0, \ \varsigma_{ij}^{(q)} > 0, \qquad i, j = 1, 2, ..., n \ (i \neq j). \tag{20c}$$

The matrix $\mathbf{\Lambda}^{(q)} = [\Lambda_{ij}^{(q)}] \in \boldsymbol{R}^{n\times n}$ $(q = 1, 2)$ *becomes a positive scalar for* $n = 1$.

Remark 7 *In Theorem 6, a new stability criterion is proposed for externally interfered discrete-time systems with saturation nonlinearities and time-varying delay. The optimal* H_∞ *performance norm bound* γ *can be obtained by solving the minimization problem subject to Theorem 6 constraints.*

Remark 8 *The proposed stability condition may be solved using YALMIP parser [20] with MATLAB. By solving LMIs of Theorem 6, the allowable maximum delay bound (AMDB) may be calculated with predefined convergence rate* α. *In addition, the positive scalar* α *may be considered as a tuning parameter to get AMDB when solving the LMIs established in Theorem 6.*

Pertaining to the system characterized by (16) and (2) with zero state-delay, the following result is presented.

Theorem 7 *For given convergence rate* $\alpha > 0$ *and attenuation level* $\gamma > 0$, *if there exist matrices* $\mathbf{\Lambda}^{(1)} = [\Lambda_{ij}^{(1)}] \in \boldsymbol{R}^{n\times n}$, $\mathbf{\Lambda}^{(2)} = [\Lambda_{ij}^{(2)}] \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{P} > \mathbf{0}$ *and* $\boldsymbol{S} > \mathbf{0}$ *such that*

$$\tilde{\mathbf{\Omega}} = \tilde{\mathbf{\Omega}}_1 + \tilde{\mathbf{\Omega}}_2 + \tilde{\mathbf{\Omega}}_2^T < \mathbf{0}, \tag{21}$$

where

$$\begin{aligned}
\tilde{\mathbf{\Omega}}_1 &= \tilde{\boldsymbol{g}}_1^T \boldsymbol{P} \tilde{\boldsymbol{g}}_1 + \tilde{\boldsymbol{g}}_1^T \boldsymbol{P} \tilde{\boldsymbol{g}}_3 + \tilde{\boldsymbol{g}}_3^T \boldsymbol{P} \tilde{\boldsymbol{g}}_1 + \tilde{\boldsymbol{g}}_3^T \boldsymbol{P} \tilde{\boldsymbol{g}}_3 - \gamma^2 \boldsymbol{g}_6^T \boldsymbol{g}_6 - \tilde{\boldsymbol{g}}_2^T (\boldsymbol{P} - \boldsymbol{S}) \tilde{\boldsymbol{g}}_2 - \tilde{\boldsymbol{g}}_1^T \left(\mathbf{\Lambda}^{(1)^T} + \mathbf{\Lambda}^{(1)} \right) \tilde{\boldsymbol{g}}_1 \\
&\quad + (e^\alpha \boldsymbol{A} \tilde{\boldsymbol{g}}_2)^T \left(\mathbf{\Lambda}^{(1)^T} + \mathbf{\Lambda}^{(1)} \right) (e^\alpha \boldsymbol{A} \tilde{\boldsymbol{g}}_2) \\
\tilde{\mathbf{\Omega}}_2 &= \tilde{\boldsymbol{g}}_1^T \mathbf{\Lambda}^{(2)^T} [e^\alpha \boldsymbol{A} \tilde{\boldsymbol{g}}_2 - \tilde{\boldsymbol{g}}_1] \\
\tilde{\boldsymbol{g}}_i &= [\mathbf{0}_{n\times(i-1)n} \quad \boldsymbol{I}_n \quad \mathbf{0}_{n\times(3-i)}] \ (i = 1, 2, 3),
\end{aligned}$$

then the system (16) *employing* (2) *with zero state-delay is exponentially stable.*

Remark 9 *In Theorem 7, a stability criterion is proposed for the externally interfered digital filters with saturation nonlinearities. The established condition ensures* H_∞ *performance against the interference and guarantees exponential stability of the system with null external disturbance.*

To verify the merit and the efficiency of the proposed theorem (Theorem 6), the interfered discrete-time system with time-varying delay and saturation nonlinearities is considered with

$$\boldsymbol{A} = \begin{bmatrix} 0.4 & 0 \\ 0.05 & 0.2 \end{bmatrix}, \ \boldsymbol{A}_d = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.1 \end{bmatrix} \text{ and } \boldsymbol{g}(k) = \begin{bmatrix} cos(k) \\ sin(k) \end{bmatrix}. \tag{23}$$

It is found that the system (16) using (2) is exponentially stable with prescribed attenuation level γ.

4.4 Stability and Performance Analysis of Externally Interfered Time-varying State-delayed Digital Filters via Generalized Overflow Nonlinearities

The methods developed in the previous sections are not suitable to validate stability of digital filters with generalized overflow nonlinearities. So, in this work, an approach is developed to examine stability of digital filters with generalized overflow nonlinearities. First, lemmas are established to characterize the generalized overflow nonlinearities in greater detail. Next, a sufficient condition is derived under which the digital filter with external disturbance, generalized overflow nonlinearities and time-varying delay is stable and has a prescribed noise attenuation level. Then, with the help of developed lemmas and a Lyapunov functional, an asymptotic stability condition is proposed for the digital filters with time-varying delay and overflow nonlinearities.

4.4.1 System Model and Problem Formulation

The discrete-time system

$$\boldsymbol{x}(k+1) = \boldsymbol{f}\big(\boldsymbol{y}(k)\big) + \boldsymbol{w}(k), \tag{24a}$$
$$\boldsymbol{y}(k) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{A}_d\boldsymbol{x}\big(k - h(k)\big), \tag{24b}$$
$$\boldsymbol{x}(k) = \boldsymbol{\phi}(k), \quad k \in [-h_2, 0], \tag{24c}$$

with the function $\boldsymbol{f}(.)$ representing generalized overflow nonlinearities [21]

$$\begin{aligned} -1 \leq q_i \leq f_i\left(y_i(k)\right) \leq q_{1i} \leq 1, &\qquad y_i(k) > 1 \\ f_i\left(y_i(k)\right) = y_i(k), &\qquad -1 \leq y_i(k) \leq 1 \\ -1 \leq -q_{2i} \leq f_i\left(y_i(k)\right) \leq -q_i \leq 1, &\qquad y_i(k) < -1 \end{aligned} \tag{25}$$

is under consideration, where q_i, q_{1i}, and q_{2i} $(i = 1, 2, ..., n)$ are known scalars.

Remark 10 *The nonlinear function (25) may be used to represent different overflow arithmetics such as two's complement* ($q_i = -1$, $q_{1i} = q_{2i} = 0$), *zeroing* ($q_i = q_{1i} = q_{2i} = 0$), *saturation* ($q_i = q_{1i} = q_{2i} = 1$) *and triangular* ($q_i = 0$, $q_{1i} = q_{2i} = 1$).

The following lemmas are used to obtain main results of this work.

Lemma 3 *For the digital filter (24)-(25) with null external disturbance, if there exist a matrix* $\boldsymbol{M} = [m_{ij}] \in \boldsymbol{R}^{n\times n}$, *diagonal matrices* $\boldsymbol{N} = [n_{ii}] \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{Q} = [q_{ii}] \in \boldsymbol{R}^{n\times n}$ *and* $\boldsymbol{U} = [u_{ii}] \in \boldsymbol{R}^{n\times n}$ *such that*

$$m_{ji} = 0, \quad i \in \boldsymbol{Z}^+ \cap (\mu, n], \tag{26a}$$
$$m_{ii} \geq |k_{ii}| + |q_{ii}| + |u_{ii}|, \quad i \in \boldsymbol{Z}^+ \cap (\mu, n], \tag{26b}$$

then the following condition holds

$$\left[\boldsymbol{y}^T(k)\boldsymbol{M} + \boldsymbol{f}^T\big(\boldsymbol{y}(k)\big)\boldsymbol{N} + \boldsymbol{x}^T(k)\boldsymbol{Q} + \boldsymbol{x}^T\big(k-h(k)\big)\boldsymbol{U}\right]\left[\boldsymbol{y}(k) - \boldsymbol{f}\big(\boldsymbol{y}(k)\big)\right] \geq 0. \tag{27}$$

Lemma 4 *For a diagonal matrix* $\boldsymbol{W} = [w_{ii}] \in \boldsymbol{R}^{n\times n}$ *and the function satisfying (25) such that*

$$w_{ii} > 0, \quad i \in \boldsymbol{Z}^+ \cap (\mu, n] \tag{28}$$

then following condition holds

$$\left[\boldsymbol{f}\big(\boldsymbol{y}(k)\big) - \boldsymbol{\Gamma}\boldsymbol{y}(k)\right]^T \boldsymbol{W}\left[\boldsymbol{y}(k) - \boldsymbol{f}\big(\boldsymbol{y}(k)\big)\right] \geq 0 \tag{29}$$

where $\boldsymbol{\Gamma} = diag\{\hat{q}_1, \hat{q}_2, ..., \hat{q}_n\}$ *and* $\hat{q}_i = min\{q_i, 0\}$.

In the following subsection, stability results for the digital filter (24)-(25) are reported with the help of Lemmas 3 and 4.

4.4.2 Main Results

A stability condition for H_∞ performance of the digital filter (24)-(25) is established as follows.

Theorem 8 *For given scalars* h_1 *and* h_2*, if there exist matrices* $\boldsymbol{P} > \boldsymbol{0}$*,* $\boldsymbol{S} > \boldsymbol{0}$*,* $\boldsymbol{\Theta}_1 > \boldsymbol{0}$*,* $\boldsymbol{\Theta}_2$*,* $\boldsymbol{\Theta}_3$*,* $\boldsymbol{R}_i > \boldsymbol{0}$ $(i = 1, 2)$*, diagonal matrices* $\boldsymbol{M}$*,* $\boldsymbol{U}$ *and a matrix* $\boldsymbol{G}$ *such that*

$$\boldsymbol{\Theta}_2 + h_{12}\boldsymbol{\Theta}_1 > \boldsymbol{0}, \quad \boldsymbol{\Theta}_3 + \boldsymbol{\Theta}_1 > \boldsymbol{0}, \tag{30}$$

$$\boldsymbol{\Upsilon} = \begin{bmatrix} \boldsymbol{R}_2 & \boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{R}_2 \end{bmatrix} \geq \boldsymbol{0}, \tag{31}$$

$$\widetilde{\boldsymbol{\Psi}} = \widetilde{\boldsymbol{\Psi}}_1 + \widetilde{\boldsymbol{\Psi}}_2 + \widetilde{\boldsymbol{\Psi}}_3 + \widetilde{\boldsymbol{\Psi}}_3^T + \widetilde{\boldsymbol{\Psi}}_4 < \boldsymbol{0}, \tag{32}$$

where

$$\begin{aligned}
\widetilde{\boldsymbol{\Psi}}_1 &= \boldsymbol{g}_1^T \boldsymbol{P} \boldsymbol{g}_1 + \boldsymbol{g}_1^T \boldsymbol{P} \boldsymbol{g}_6 + \boldsymbol{g}_6^T \boldsymbol{P} \boldsymbol{g}_1 + \boldsymbol{g}_6^T \boldsymbol{P} \boldsymbol{g}_6 - \gamma^2 \boldsymbol{g}_6^T \boldsymbol{g}_6 \\
&\quad + \boldsymbol{g}_2^T \left[(h_{12} + 2) * \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2 + \boldsymbol{\Theta}_3 - \boldsymbol{P} + \boldsymbol{S}\right] \boldsymbol{g}_2 - \boldsymbol{g}_3^T (\boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2) \boldsymbol{g}_3 \\
&\quad - \boldsymbol{g}_4^T \boldsymbol{\Theta}_1 \boldsymbol{g}_4 - \boldsymbol{g}_5^T (\boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_3) \boldsymbol{g}_5 + \widetilde{\boldsymbol{J}}_0^T (h_1^2 \boldsymbol{R}_1 + h_{12}^2 \boldsymbol{R}_2) \widetilde{\boldsymbol{J}}_0 \\
\widetilde{\boldsymbol{\Psi}}_2 &= -\boldsymbol{J}_1^T \boldsymbol{R}_1 \boldsymbol{J}_1 - \begin{bmatrix} \boldsymbol{J}_2^T & \boldsymbol{J}_3^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\Upsilon} \end{bmatrix} \begin{bmatrix} \boldsymbol{J}_2^T & \boldsymbol{J}_3^T \end{bmatrix}^T \\
\widetilde{\boldsymbol{\Psi}}_3 &= [\boldsymbol{g}_1 - \boldsymbol{\Gamma}(\boldsymbol{A}\boldsymbol{g}_2 + \boldsymbol{A}_d \boldsymbol{g}_4)]^T \boldsymbol{U} [\boldsymbol{A}\boldsymbol{g}_2 + \boldsymbol{A}_d \boldsymbol{g}_4 - \boldsymbol{g}_1] \\
\widetilde{\boldsymbol{\Psi}}_4 &= [(\boldsymbol{A}\boldsymbol{g}_2 + \boldsymbol{A}_d \boldsymbol{g}_4)^T \boldsymbol{M} (\boldsymbol{A}\boldsymbol{g}_2 + \boldsymbol{A}_d \boldsymbol{g}_4) - \boldsymbol{g}_1^T \boldsymbol{M} \boldsymbol{g}_1] \\
\widetilde{\boldsymbol{J}}_0 &= \boldsymbol{g}_1 + \boldsymbol{g}_6 - \boldsymbol{g}_2, \quad \boldsymbol{J}_i = \boldsymbol{g}_{i+1} - \boldsymbol{g}_{i+2} \ (i = 1, 2, 3) \\
\boldsymbol{g}_i &= [\boldsymbol{0}_{n \times (i-1)n}, \boldsymbol{I}_n, \boldsymbol{0}_{n \times (6-i)n}] \ (i = 1, 2, ..., 6) \\
h_{12} &= h_2 - h_1,
\end{aligned}$$

then the system (24) and (25) is asymptotically stable and has H_∞ *norm less than or equal to* γ *from* $\boldsymbol{w}(k)$ *to* $\boldsymbol{x}(k)$.

Remark 11 *A stability condition for overflow oscillations free externally disturbed discrete-time systems with state-delay and overflow nonlinearities is given in Theorem 8. By solving the stability condition which is characterized by LMIs, AMDB may be calculated with predefined* H_∞ *norm* γ.

Next, an improved stability condition is proposed for the digital filters with time-varying delay and overflow nonlinearities.

Theorem 9 *For given scalars* h_1 *and* h_2*, if there exist symmetric matrices* $\boldsymbol{P} > \boldsymbol{0}$*,* $\boldsymbol{\Theta}_1 > \boldsymbol{0}$*,* $\boldsymbol{\Theta}_2$*,* $\boldsymbol{\Theta}_3$*,* $\boldsymbol{R}_i > \boldsymbol{0}$ $(i = 1, 2)$*, a matrix* $\boldsymbol{M}$ *satisfying* (26)*, diagonal matrices* $\boldsymbol{N}$*,* $\boldsymbol{Q}$*,* $\boldsymbol{U}$*,* $\boldsymbol{W}$ *and a matrix* $\boldsymbol{G}$ *such that* (28) *holds and*

$$\boldsymbol{\Theta}_2 + h_{12}\boldsymbol{\Theta}_1 > \boldsymbol{0}, \quad \boldsymbol{\Theta}_3 + \boldsymbol{\Theta}_1 > \boldsymbol{0}, \tag{33}$$

$$\boldsymbol{\Upsilon} = \begin{bmatrix} \boldsymbol{R}_2 & \boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{R}_2 \end{bmatrix} \geq \boldsymbol{0}, \tag{34}$$

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}_1 + \boldsymbol{\Psi}_2 + \boldsymbol{\Psi}_3 + \boldsymbol{\Psi}_3^T + \boldsymbol{\Psi}_4 + \boldsymbol{\Psi}_4^T < \boldsymbol{0}, \tag{35}$$

where

$$\begin{aligned}
\boldsymbol{\Psi}_1 &= \boldsymbol{g}_1^T \boldsymbol{P} \boldsymbol{g}_1 + \boldsymbol{g}_2^T \left[(h_{12}+2) * \boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2 + \boldsymbol{\Theta}_3 - \boldsymbol{P} \right] \boldsymbol{g}_2 - \boldsymbol{g}_3^T (\boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_2) \boldsymbol{g}_3 \\
&\quad - \boldsymbol{g}_4^T \boldsymbol{\Theta}_1 \boldsymbol{g}_4 - \boldsymbol{g}_5^T (\boldsymbol{\Theta}_1 + \boldsymbol{\Theta}_3) \boldsymbol{g}_5 + \boldsymbol{J}_0^T (h_1^2 \boldsymbol{R}_1 + h_{12}^2 \boldsymbol{R}_2) \boldsymbol{J}_0 \\
\boldsymbol{\Psi}_2 &= -\boldsymbol{J}_1^T \boldsymbol{R}_1 \boldsymbol{J}_1 - \left[\boldsymbol{J}_2^T \ \ \boldsymbol{J}_3^T \right] \left[\boldsymbol{\Upsilon} \right] \left[\boldsymbol{J}_2^T \ \ \boldsymbol{J}_3^T \right]^T \\
\boldsymbol{\Psi}_3 &= [\boldsymbol{g}_1 - \boldsymbol{\Gamma}(\boldsymbol{A}\boldsymbol{e}_2 + \boldsymbol{A}_d \boldsymbol{g}_4)]^T \boldsymbol{W} [\boldsymbol{A}\boldsymbol{e}_2 + \boldsymbol{A}_d \boldsymbol{g}_4 - \boldsymbol{g}_1] \\
\boldsymbol{\Psi}_4 &= [(\boldsymbol{A}\boldsymbol{e}_2 + \boldsymbol{A}_d \boldsymbol{g}_4)^T \boldsymbol{M} + \boldsymbol{g}_1^T \boldsymbol{N} + \boldsymbol{g}_2^T \boldsymbol{Q} + \boldsymbol{g}_4^T \boldsymbol{U}][\boldsymbol{A}\boldsymbol{e}_2 + \boldsymbol{A}_d \boldsymbol{g}_4 - \boldsymbol{g}_1] \\
\boldsymbol{J}_i &= \boldsymbol{g}_{i+1} - \boldsymbol{g}_{i+2} \ \ (i = 0, 1, 2, 3) \\
\boldsymbol{g}_i &= [\boldsymbol{0}_{n \times (i-1)n}, \boldsymbol{I}_n, \boldsymbol{0}_{n \times (5-i)n}] \ \ (i = 1, 2, ..., 5) \\
h_{12} &= h_2 - h_1,
\end{aligned}$$

then the system (24)-(25) is asymptotically stable.

Remark 12 *Theorem 9 provides a more relaxed stability criterion with reduced computational burden as compared to the existing conditions [21–23]. This improvement is achieved due to the application of a better Lyapunov functional and newly developed lemmas (3 and 4). Theorem 9 is in LMI settings and so, it may be solved using YALMIP parser [20] with MATLAB.*

Table 1: The AMDB for C_1

	h_1				
Methods	3	5	10	30	N_d
Theorem 1 [21]	4	6	11	31	38
Theorem 2 [23]	4	6	11	31	44
Theorem 1 [22]	8	10	15	35	39
Theorem 1	9	11	16	36	33

Table 2: The AMDB for C_2

	h_1				
Methods	3	5	10	30	N_d
Theorem 1 [21]	10	12	17	37	38
Theorem 2 [23]	11	13	18	38	44
Theorem 1 [22]	14	16	21	41	39
Theorem 1	15	17	22	42	33

A comparative study of the proposed approach with the existing conditions is made on the basis of delay range to highlight usefulness of the established criterion. Consider the digital filter (24)-(25) with

$$\boldsymbol{A} = \begin{bmatrix} 0.3 & -0.4 \\ 0.5 & 0.7 \end{bmatrix}, \ \boldsymbol{A}_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ \ \boldsymbol{g}(k) = \boldsymbol{0} \tag{36}$$

and overflow arithmetic cases:

$C_1 : q_1 = 0, q_2 = -1, \ q_{i1} = 1, \ q_{i2} = 0 \ (i = 1, 2), \qquad C_2 : q_1 = -1, q_2 = 0, \ q_{ij} = 0 \ (i, j = 1, 2).$

For different values h_1, Tables 1 (for C_1) and 2 (for C_2) list AMDB produced by Theorem 9 and other existing conditions [21–23] such that system (24)-(25) with null disturbance is asymptotically stable. It is clear from the tables that the proposed approach is less stringent than the existing conditions [21–23].

5 Conclusions and Scope for Future Work

5.1 Conclusions

In this thesis, stability of digital filters with overflow nonlinearities, external disturbance and state-delay has been investigated using H_∞ based approach. The following are the key highlights of the proposed work.

- Stability of discrete-time state-delayed systems with saturation nonlinearities and external disturbance has been investigated. A criterion has been reported to ensure exponential stability of the system without external disturbance and guaranteed attenuation level in the presence of the disturbance. Further, an improved stability condition has been developed for the discrete-time state-delayed systems employing saturation arithmetic and experiencing disturbance by relaxing constraints on saturation characterization. A numerical example has been given to validate the proposed work.

- An approach to develop a better stability criterion than existing criteria has been carried out by making use of better sector-based characterization. As a result of better characterization of saturation overflow arithmetic, a stability criterion has been established which leads to better stability region and better estimate of H_∞ performance norm. In addition to that, a less stringent asymptotic stability has been proposed for the digital filters employing saturation arithmetic. It has been shown that many existing conditions can be achieved from the established condition.

- A new H_∞ stability approach for discrete-time systems with time-varying delay, external disturbance and saturation nonlinearities has been developed. By constructing a proper Lyapunov functional and system transformation, a delay-dependent stability criterion has been framed for the fixed-point digital filters with time-varying delay, saturation nonlinearities and external interference. The reported criterion not only promises guaranteed H_∞ performance but also ensures exponential stability of the digital filters in the absence of the external disturbance. Further, a stability condition has been developed for the interference free digital filters with saturation arithmetic and demonstrated to be less conservative than the existing results.

- Asymptotic stability and H_∞ performance of interfered digital filters with time-varying and generalized overflow nonlinearities which include saturation, zeroing, triangular and two's complement have been investigated using new characterization of overflow arithmetics. Two lemmas related to characterization of overflow nonlinearities have been established with better utilization of system information. The developed lemmas use less number of decision variables as compared to exiting results in the literature. With the help of developed lemmas and a Lyapunov functional that adds more information about delayed states of the system, a stability condition has been proposed for the digital filters with time-varying delay and overflow nonlinearities. The presented condition has been shown to be more relaxed and computationally less demanding than the existing criteria. Further, effects of external disturbance in the digital filters with time-varying delay and generalized overflow nonlinearities have been reduced to H_∞ performance norm level.

5.2 Future Work

In this thesis, many conditions have been proposed to guarantee stability of the discrete-time systems with overflow nonlinearities. The possible future research works are listed as follows:

- It is well known that time delay are commonly encountered in digital filters due to computational or transportation delay. The presence of time-delay may have effects on stability of the digital filters and so, many results have been reported for digital filters with time-delay in this thesis. However, the existing conditions are developed in the global context and no condition is reported in the local context [24]. So, examination of stability of digital filters with time-delay in the local stability context seems to be appealing future work.

- It may be noticed that the results reported in this thesis have not considered the quantization effects under decoupling assumption [25]. However, a few research works have been developed to investigate stability of digital filters with both overflow and quantization nonlinearities [26, 27]. The possible extension of proposed H_∞ approach for the investigation of stability of digital filters with combined effects of quantization and overflow will be a more realistic problem for further investigation.

- The proposed approach provide stability conditions for digital filters with single time-varying delay. The established techniques can be further explored to establish stability criterion for the digital filters with overflow nonlinearities and multiple time-varying delays [28].
- Further, the possible extension of presented approaches to the problems of Hankel norm performance of digital filters [19], dissipativity analysis of digital filters [15] and stability investigation of digital filters with Markovian jumping parameters [16] demands further investigation.

6 Proposed Contents of the Thesis

The organization of the thesis is as follows:

- **Chapter 1** Introduction
- **Chapter 2** Realization of Overflow Oscillations Free Interfered Nonlinear Discrete-time Systems with State-delay and Saturation Nonlinearities
- **Chapter 3** Stability Investigation of Externally Disturbed State-delayed Digital Filters Employing Saturation Arithmetic
- **Chapter 4** Delay-dependent Stability Criteria for Interfered Digital Filters with Time-varying Delay and Saturation Nonlinearities
- **Chapter 5** Stability and Performance Analysis of Externally Interfered Time-varying State-delayed Digital Filters via Generalized Overflow Nonlinearities
- **Chapter 6** Conclusions and Scope for Future Work

7 Publications

7.1 Part of Thesis

7.1.1 Journals

1. Parthipan C G and P Kokil, "Delay-dependent stability analysis of interfered digital filters with time-varying delay and saturation nonlinearities," *Circuits, Systems, and Signal Processing* (Minor revision)
2. Parthipan C G and P Kokil, "Stability of state-delayed digital filters with overflow nonlinearities," *Transactions of the Institute of Measurement and Control* (Accepted for publication)
3. Parthipan C G and P Kokil, "Stability of digital filters with state-delay and external interference," *Circuits, Systems, and Signal Processing*, vol. 40, no. 8, pp. 3866-3883, 2021.
4. Parthipan C G and P Kokil, "Overflow oscillations free implementation of state-delayed digital filter with saturation arithmetic and external disturbance," *Transactions of the Institute of Measurement and Control*, vol. 42, no. 2, pp. 188-197, 2020.
5. P Kokil, Parthipan C G, S Jogi and H Kar, "Criterion for realizing state-delayed digital filters subjected to external interference employing saturation arithmetic," *Cluster Computing*, vol. 22, no. 6, pp. 15187-15194, 2019.

7.2 Other Contributions

7.2.1 Journals

1. P Kokil and Parthipan C G, "Stability of digital filters subject to external interference and state-delay," *Transactions of the Institute of Measurement and Control*, vol. 42, no. 13, pp. 2559-2568, 2020.
2. S Jogi, Parthipan C G, P Kokil, "A passivity based approach for digital filters subjected to external disturbance and nonlinearities," *IFAC-PapersOnLine*, vol. 53, no. 1, pp. 428-434, 2020.
3. Parthipan C G, X S Arockiaraj and P Kokil, "New passivity results for the realization of interfered digital filters utilizing saturation overflow nonlinearities," *Transactions of the Institute of Measurement and Control*, vol.40, no.15, pp. 4246-4252, 2018.

7.2.2 Conferences

1. Parthipan C G and P Kokil, "Realization of limit-cycle free digital filters with improved characterization of saturation arithmetic," in Proceedings of *IEEE Region 10 Symposium* (TENSYMP), Jeju, Republic of Korea, August 2021.
2. Parthipan C G and P Kokil, "Limit-cycle free realization of digital filters," in Proceedings of *International Conference on Signal Processing and Communications* (SPCOM), IISc Bangalore, India, July 2020.
3. P Kokil, Parthipan C G and H Kar, "Stability of state-delayed digital filters with saturation arithmetic and external disturbance," in Proceedings of *Conference on Nonlinear Systems and Dynamics (CNSD)*, IIT Kanpur, India, December 2019.

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